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TECHNICAL NOTES

Laminar heat transfer in a tube with viscous dissipation

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NOMENCLATURE

c_p	specific heat at constant pressure
c_n	eigen coefficient
Ec	Eckert number, $\bar{u}^2/c_p(t_i - t_w)$ or $\bar{u}^2 k/c_p \bar{q} r_0$
h	local film heat transfer coefficient
k	thermal conductivity of fluid
M	product of Eckert and Prandtl numbers, $\mu\bar{u}^2/k(t_i - t_w)$ for the isothermal and $\mu\bar{u}^2/\bar{q}r_0$ for the constant heat flux boundary condition
Nu_x	local Nusselt number, $2hr_0/k$
Pe	Peclet number, $2\bar{u}r_0/\alpha$
Pr	Prandtl number of fluid, $\mu c_p/k$
$\bar{q}$	constant heat flux at the boundary
r	radial coordinate
r_0	radius of the tube
r^*	dimensionless radial coordinate, r/r_0
R_n	eigen function
t	temperature of fluid
t_i	temperature of fluid at inlet
t_w	temperature of fluid at the wall
t_b	bulk-mean temperature of fluid
t^*	dimensionless temperature of fluid, $(t - t_w)/(t_i - t_w)$, for the isothermal and $k(t - t_i)/\bar{q}r_0$ for the constant heat flux boundary condition
u	velocity of fluid, $2\bar{u}(1 - r^2/r_0^2)$
$\bar{u}$	average velocity of fluid
x	axial coordinate
x^*	dimensionless axial coordinate, $x/Pe\, r_0$.

Greek symbols

ρ	fluid density
α	thermal diffusivity, $k/c_p\rho$
λ_n	eigen value
μ	dynamic viscosity of fluid.

1. INTRODUCTION

THE CLASSICAL Graetz problem of laminar heat transfer through a circular tube, neglecting axial heat conduction and viscous dissipation, has been worked out in detail by Sellars *et al.* [1]. Recently, a number of papers have been published on the modified version of the Graetz problem by including the effect of axial heat conduction. The effect of axial heat conduction becomes important for fluids having low Prandtl number. The inclusion of the axial heat conduction effect in the Graetz problem brings in some complexities in the solution of the problem which have been tackled suitably in a number of ways, including that of Nagasue [2].

The present paper is another modification of the Graetz problem which takes into account viscous dissipation but neglects the effect of axial conduction. The problem seems to be important for fluids having large Prandtl number. In this case, the axial heat conduction effect is negligible and the hydrodynamic entrance length is rather small compared to that of thermal. Consequently, one may neglect the effect of viscous dissipation in the initial hydrodynamic developing region and approximate the temperature profile at the inlet of the thermal heating or cooling region by an average constant value. Unlike the case of inclusion of axial heat conduction, the solution for the present problem is found to be rather simpler.

The differential equation for the problem can be written as

$$2\rho c_p \bar{u}(1 - r^2/r_0^2)\frac{\partial t}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \mu\left(\frac{\partial u}{\partial r}\right)^2. \quad (1)$$

The initial and boundary conditions are given by

$$x = 0: \quad t = t_i$$

$$x > 0: \quad t(x, r_0) = t_w \quad \text{or} \quad k\frac{\partial t}{\partial r}(x, r_0) = \bar{q}.$$

The dimensionless form of equation (1) reduces to

$$(1 - r^{*2})\frac{\partial t^*}{\partial x^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial t^*}{\partial r^*}\right) + 16Mr^{*2}. \quad (2)$$

The initial and boundary conditions in dimensionless form become for

(i) *isothermal case*

$$x^* = 0: \quad t^* = 1$$

$$x^* > 0: \quad t^* = 0 \quad \text{at} \quad r^* = 1$$

(ii) *constant heat flux case*

$$x^* = 0: \quad t^* = 0$$

$$x^* > 0: \quad \frac{\partial t^*}{\partial r^*} = 1 \quad \text{at} \quad r^* = 1.$$

2. SOLUTION FOR ISOTHERMAL BOUNDARY CONDITION

The solution of temperature for this boundary condition can be expressed in the form

$$t^* = M(1 - r^{*4}) + \sum_{n=1}^{\infty} c_n R_n(r^*, \lambda_n)\exp(-\lambda_n^2 x^*) \quad (3)$$

where λ_n satisfies the differential equation

$$r^{*2}R_n'' + r^* R_n' + \lambda_n^2(r^{*2} - r^{*4})R_n = 0 \quad (4)$$

subject to the boundary conditions $R_n(1) = 0$ and $R_n(0) = 1$. It may be noted that the eigen values and eigen functions of equation (4) are identical to Graetz's problem, which have been discussed in detail by Sellars *et al.* [1]. The eigen constants c_n of equation (4) are, however, different from Graetz's original problem for isothermal boundary conditions and can be obtained from the relationship

$$c_n = \frac{\int_0^1 [1 - M(1 - r^{*4})]R_n r^*(1 - r^{*2})\,dr^*}{\int_0^1 R_n^2 r^*(1 - r^{*2})\,dr^*} \quad (5)$$

Table 1. Eigen constants of equation (6)

M	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0.000	1.47643	−0.80612	0.58876	−0.47585	0.40502	−0.35577	0.31922
0.001	1.47508	−0.80562	0.58853	−0.47572	0.40493	−0.35511	0.31917
0.100	1.34109	−0.75572	0.56554	−0.46261	0.39648	−0.34983	0.31484
1.000	0.12302	−0.30209	0.35658	−0.34349	0.31964	−0.29641	0.27546
5.000	−5.29062	1.71404	−0.57214	0.18595	−0.02188	−0.05899	0.10046
10.000	−12.05767	4.23421	−1.73306	0.84775	−0.44879	0.23776	−0.11830

The details of calculations of the eigen constants c_n are reported in reference [3]. The first five eigen constants have been calculated using the series solution for R_n and the higher ones through the WKB approximation method. Table 1 gives a few values of eigen constants for several values of M.

The local Nusselt number is found on the difference of bulk temperature and wall temperature. We have the following relations

$$-k\frac{\partial t}{\partial r}\bigg|_{r=r_0} = h(t_b - t_w) \quad Nu_x = -2\frac{\partial t^*}{\partial r^*}\bigg|_{r^*=1}\bigg/ t_b^*$$

where

$$t_b^* = \frac{\int_0^1 t^*(1-r^{*2})r^*\,dr^*}{\int_0^1 (1-r^{*2})r^*\,dr^*}.$$

It may be shown after some mathematical manipulations that

$$Nu_x = \frac{\sum_{n=1}^{\infty} c_n R_n'(1)\exp(-\lambda_n^2 x^*) - 4M}{2\left[\sum_{n=1}^{\infty} \frac{c_n R_{n(1)}'}{\lambda_n^2}\exp(-\lambda_n^2 x^*) - \frac{5M}{24}\right]}. \quad (6)$$

3. SOLUTION FOR CONSTANT HEAT FLUX BOUNDARY CONDITION

The solution of temperature for this boundary condition can be written in the form

$$t^* = -\frac{7}{24} - M - Mr^{*4} + (4+16M)\left(x^* + \frac{r^{*2}}{4} - \frac{r^{*4}}{16}\right) + \sum_{n=1}^{\infty} c_n R_n(r^*, \lambda_n)\exp(-\lambda_n^2 x^*) \quad (7)$$

in which λ_n satisfies the differential equation (4), subject to the boundary conditions $R_n'(1) = 0$ and $R_n(0) = 1$. The eigen functions R_n and eigen constants λ_n of equation (7) are identical to the solution of the problem without viscous dissipation reported earlier by Siegel *et al.* [4]. The eigen constants are, however, different and given by the relation

$$c_n = -\frac{\int_0^1 \left(r^{*2} - \frac{r^{*4}}{4} + 4Mr^{*2} - 2Mr^{*4}\right)(1-r^{*2})r^* R_n\,dr^*}{\int_0^1 R_n^2(1-r^{*2})r^*\,dr^*} \quad (8)$$

Table 2 gives a few values of eigen constants for several values of M.

The local Nusselt number is evaluated on the difference of bulk temperature and wall temperature. We have the following relations

$$\bar{q} = h(t_w - t_b) \quad Nu_x = \frac{2}{(t_w^* - t_b^*)} \quad t_b^* = (1+4M)4x^*$$

$$t_w^* - t_b^* = M + \frac{11}{24} + \sum_{n=1}^{\infty} c_n R_n(1)\exp(-\lambda_n^2 x^*)$$

whence

$$Nu_x = \frac{2}{M + \frac{11}{24} + \sum_{n=1}^{\infty} c_n R_n(1)\exp(-\lambda_n^2 x^*)}. \quad (9)$$

4. RESULTS AND DISCUSSION

To calculate the local Nusselt number according to equations (6) and (9), we require to know the eigen values and eigen functions or its derivatives at the wall, apart from the values of eigen constants reported in Tables 1 and 2. For ready reference, the first seven values of the relevant parameters are given in Table 3 adopted from references [1] and [4].

A plot of equation (6) for several values of M, the product of Eckert and Prandtl numbers, is shown in Fig. 1. For a particular Graetz number the local Nusselt number is found to increase with the increase in the value of the parameter M. For all values of M, other than zero, the asymptotic Nusselt number attains the value 9.6. The asymptotic Nusselt number 9.6 can easily be derived from equation (6) because the exponential terms disappear at large distances from the entrance. The interesting feature of the result obtained from the present analysis is that the asymptotic value of Nusselt number for isothermal boundary condition is considerably higher than that of 3.656 for the classical Graetz problem in which M is taken as zero.

In Fig. 2 is shown the plot of equation (9) for several values of the product of Eckert and Prandtl numbers. Unlike the isothermal case, the local Nusselt number decreases with the increase in values of M. It may be seen from equation (9) that the asymptotic Nusselt number is given by the expression $2/(M + 11/24)$. The asymptotic Nusselt number will, therefore, be in general lower than 4.36, the value obtained for the corresponding Graetz problem having constant heat flux at the boundary and without any viscous dissipation of heat.

Table 2. Eigen constants of equation (9)

M	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0.000	0.40348	−0.17511	0.10559	−0.07328	0.05503	−0.04348	0.03559
0.001	0.40476	−0.17551	0.10579	−0.07340	0.05511	−0.04353	0.03576
0.100	0.53141	−0.21575	0.12551	−0.08510	0.06286	−0.04905	0.03992
1.000	1.68280	−0.58158	0.30477	−0.19150	0.13332	−0.09915	0.07775
5.000	6.80007	−2.20737	1.10149	−0.66439	0.44649	−0.32184	0.24587
10.000	13.19666	−4.23984	2.09740	−1.25551	0.84194	−0.60021	0.45603

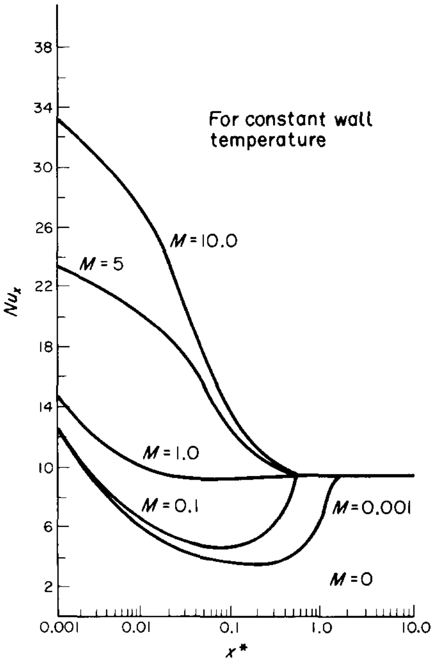

FIG. 1. Variation of local Nusselt number in laminar flow in a tube with viscous dissipation for isothermal boundary condition.

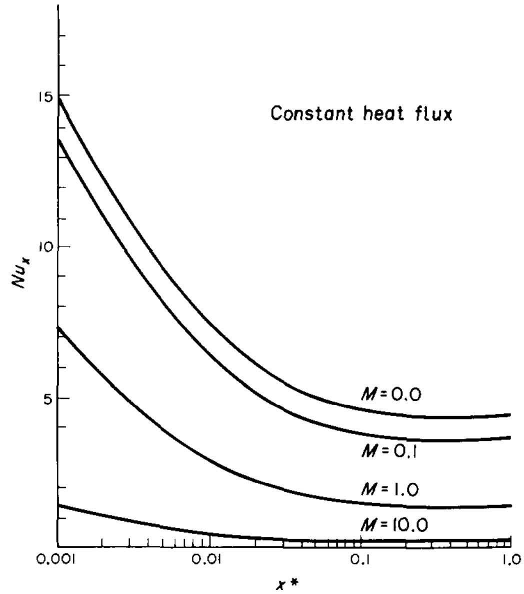

FIG. 2. Variation of local Nusselt number in laminar flow in a tube with viscous dissipation for constant heat flux boundary condition.

Table 3. Eigen values and eigen functions of equations (6) and (9)

Isothermal boundary		Constant heat flux boundary	
λ_n^2	$R_n'(1)$	λ_n^2	$R_n(1)$
7.3136	−1.01430	25.6796	−0.49251
44.6095	1.34924	83.8618	0.39551
113.9210	−1.57231	174.1670	−0.34587
215.2406	1.74600	296.5360	0.31405
348.5640	−1.89085	450.9477	−0.29125
513.7777	2.01450	637.3877	0.27381
711.1111	−2.12594	855.8506	−0.25985

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Analysis of three-dimensional solidification interface shape

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NOMENCLATURE

- A_1, A_2 shape coefficients in functions F and G
- B_1, B_2 amplitude coefficients in functions F and G
- f, g arbitrary functions, $F = f/K$, $G = g/K$
- J, M, N, P, R functions defined in equations (9) and (10)
- K a constant
- k thermal conductivity of solidified material
- l half-wavelength of heating variation in y direction, $L = l/w$
- n normal to solidification interface, $N = n/w$
- q heat flow rate per unit area
- t temperature
- w half-wavelength of heating variation in x direction
- x, y, z Cartesian coordinates, $X = x/w$, $Y = y/w$, $Z = z/w$.